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SOME NOTES ON HEAT-TRANSFER BY RADIATION

BY

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The following is a discussion of the heat transfer caused by radiation between two bodies of different temperatures, one of them entirely surrounding the other.

In the first part of the article we make some simplifying assumptions often allowed for technical surfaces (later on we are going to discuss the significance of the most important among them):

(1) The temperature-radiation obeys the cosine emission law.

(2) The surfaces of the bodies are reflecting according to the cosine law of reflection (completely diffuse reflection). The bodies are opaque.

(3) The reflectivity is independent of temperature and wavelength. According to Kirchhoff's law this means that the emitted temperature-radiation obeys Stefan-Boltzmann's law.

(4) The inner surface is everywhere convex and the outer one is everywhere concave.

(5) The temperature is constant on each body.

The energy emitted in unit of time from the inner body is $A_1 c_1 T_1^4$, where A_1 is the area of the surface, c_1 the "radiation-constant" and T_1 the absolute temperature of the body. If the surroundings are non-reflecting (absolutely black) the energy received and absorbed by the inner body in unit of time will be $A_1 c_1 T_2^4$, where T_2 is the absolute temperature of the surroundings. Hence, the net loss of energy from the inner body will be

$$H = A_1 c_1 (T_1^4 - T_2^4). \quad (1)$$

This formula is often used in practical calculations. It is only valid if the radiation emitted from the inner body and reabsorbed after reflection from the surroundings can be neglected.

If the reflected radiation is not vanishingly small, the loss of energy is less than given by (1). CHRISTIANSEN[1] arrived at the following formula:

$$H = \frac{A_1 c_1 (T_1^4 - T_2^4)}{1 + c_1 \left(\frac{1}{c_2} - \frac{1}{c_0}\right) \frac{A_1}{A_2}}. \tag{2}$$

Index 2 refers to the outer surface; c_0 is the radiation-constant for a black body (Stefan's constant).

CLAUSING[2] and SAUNDERS[3] have shown, that (2) is not always correct; H also to some extent depends on the form and mutual position of the two surfaces. SAUNDERS has shown how to make corrections for this dependence if the reflectivity is so small that it is sufficient to take into account only one reflection.

We shall give equations determining H, show in which cases (2) is correct, and find an approximate solution in the general case.

The integral equations of the problem.

We choose two points x_1 and x_2 on the inner and outer surface, respectively (cf. fig. 1), so that x_1 can be seen from x_2 and vice versa. That part of the outer surface which can be seen from x_1 is denoted by A_2', while A_2'' means that part which cannot be seen from x_2. A_1' denotes that part of the inner surface, which can be seen from x_2. By $\varphi(x_1 x_2)$ we denote the function $\frac{\cos i_1 \cdot \cos i_2}{\pi r^2}$, where i_1 and i_2 are the angles shown in fig. 1, while r is the distance between the points x_1 and x_2. It is equal to the fraction of the radiation from the vicinity of x_1, which goes directly to unit of area around the point x_2 or vice versa. The corresponding function for radiation between two points x_2 and x_2' of the outer surface is denoted by $\varphi(x_2 x_2')$.

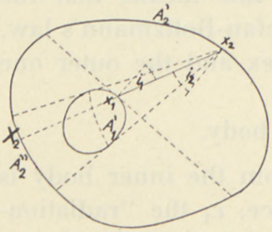

Fig. 1.

The resulting radiation (both emitted and reflected) in unit of time from unit of area near a point x is called $I(x)$.

[1] C. CHRISTIANSEN: Ann. d. Phys. u. Chem., Vol. 19, p. 267, 1883.
[2] CLAUSING: Revue d'Optique, Vol. 10, p. 353, 1931.
[3] SAUNDERS: Proc. of the Phys. Soc., Vol. 41, p. 569, 1929.

We now express that this resulting radiation is the sum of the emitted and the reflected radiation.

This leads to the following two integral equations:

$$I_1(x_1) = c_1 T_1^4 + \left(1 - \frac{c_1}{c_0}\right) \cdot \int_{A_2'} I_2(x_2)\, \varphi(x_1 x_2)\, dx_2 \tag{3}$$

$$I_2(x_2) = c_2 T_2^4 + \left(1 - \frac{c_2}{c_0}\right) \cdot \left\{ \int_{A_1'} I_1(x_1)\, \varphi(x_1 x_2)\, dx_1 + \int_{A_2 - A_2''} I_2(x_2')\, \varphi(x_2 x_2')\, dx_2' \right\}, \tag{4}$$

where as before index 1 refers to the inner, index 2 to the outer surface; dx_1, dx_2, and dx_2' denote surface elements. We have made use of Kirchhoff's law, according to which the reflectivity of a surface with the radiation-constant c is $1 - \frac{c}{c_0}$. The absorptivity is $\frac{c}{c_0}$.

Let X_2 denote the point where a straight line from x_2 to x_1 intersects the outer surface again (cf. fig. 1). It is then easily seen, that (4) can be rewritten as

$$\left.\begin{aligned} I_2(x_2) = c_2 T_2^4 + \left(1 - \frac{c_2}{c_0}\right) \cdot \Bigg\{ & \int_{A_1'} \left(I_1(x_1) - I_2(X_2)\right) \cdot \varphi(x_1 x_2)\, dx_1 + \\ & + \int_{A_2} I_2(x_2')\, \varphi(x_2 x_2')\, dx_2' \Bigg\}. \end{aligned}\right\} \tag{4 a}$$

The net energy-loss from the inner body is the difference between emitted radiation and absorbed radiation:

$$H = A_1 c_1 T_1^4 - \frac{c_1}{c_0} \cdot \int_{A_1} dx_1 \int_{A_2'} I_2(x_2)\, \varphi(x_1 x_2)\, dx_2 . \tag{5}$$

The equations (3), (4a), and (5) determine H, when the geometry of the system is known. They cannot often be solved exactly.

We first want to emphasise that Christiansen's formula (2) is valid, if the function

$$\varphi(x_2) = \int_{A_1'} \varphi(x_1 x_2)\, dx_1 \tag{6}$$

(i. e. the fraction of the radiation from the vicinity of x_2 which goes directly to the inner body) is independent of x_2. In that case $\varphi(x_2)$ can easily be found: From the definition (6) it follows that

$$\int_{A_2} \varphi(x_2)\,dx_2 = \int_{A_2} dx_2 \int_{A_1'} \varphi(x_1 x_2)\,dx_1 = \int_{A_1} dx_1 \int_{A_2'} \varphi(x_1 x_2)\,dx_2 = A_1, \quad (7)$$

so that if $\varphi(x_2)$ is constant, it must be equal to $\frac{A_1}{A_2}$.

In general $\frac{A_1}{A_2}$ is the mean value of $\varphi(x_2)$ over the outer surface.

When $\varphi(x_2) = \frac{A_1}{A_2}$, it is seen that (3) and (4 a) are satisfied by constant values of $I_1(x_1)$ and $I_2(x_2)$. Solving for I_1 and I_2 and inserting in (5), we get Christiansen's formula (2).

Some very simple forms and symmetrical arrangements of the two bodies give a constant value of $\varphi(x_2)$, e.g. two concentric spheres, two coaxial cylinders, or a sphere with a thin disk covering the equatorial plane (see below). In these cases Christiansen's formula (2) is valid, but if $\varphi(x_2)$, and consequently $I_1(x_1)$ and $I_2(x_2)$, vary, this formula is no longer correct. A simple example that can be solved exactly, will show this: Let the outer surface be a sphere and the inner body a hemisphere with slightly smaller radius (cf. fig. 2). Formula (2) then is valid for the radiation from the hemispherical surface and the plane surface separately (cf. p. 12). This means that H is a sum of two expressions of the form (2) with $\frac{A_1}{A_2}$ in the denominator replaced by 1 and $\frac{1}{2}$, respectively. This sum, however, is different from what is obtained by using formula (2) for the radiation from the total inner surface (putting $\frac{A_1}{A_2} = \frac{3}{4}$ in the denominator). One often gets a better approximation than (2) by separating the radiation in two or more parts of the form (2) as in this instance, e. g. when dealing with a flat radiator placed near a wall. On page 13 one more example is given, where such a separation is exact.

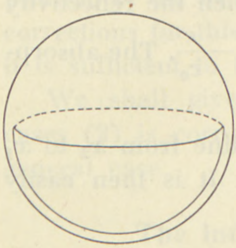

Fig. 2.

Approximate solution of the integral equations.

We shall now show how to find approximate solutions of the equations (3), (4a), and (5) in the general case.

The crudest approximation—formula (1)—is obtained by disregarding the integral $\int_{A_1'} \cdots dx_1$ in (4a); this integral represents the influence of the inner body on the radiation from the outer surface. By doing so one gets the solution (black-body radiation):

$$I_2(x_2) = c_0 T_2^4; \tag{8}$$

which inserted in (3) gives

$$I_1(x_1) = c_1(T_1^4 - T_2^4) + c_0 T_2^4. \tag{9}$$

(5) then leads to the "zero$^{\text{th}}$" approximation (1).

These expressions for I_2 and I_1 are now inserted in the integral in (4 a) which was first disregarded. In the other terms of the equations (3), (4a), and (5) we put

$$I_1(x_1) = c_1(T_1^4 - T_2^4) + c_0 T_2^4 + f(x_1) \tag{10}$$

$$I_2(x_2) = c_0 T_2^4 + g(x_2). \tag{11}$$

We then get the following equations, where we have introduced the function $\varphi(x_2)$ defined by (6):

$$f(x_1) = \left(1 - \frac{c_1}{c_0}\right) \cdot \int_{A_2'} g(x_2)\,\varphi(x_1 x_2)\,dx_2 \tag{12}$$

$$g(x_2) = \left(1 - \frac{c_2}{c_0}\right) \cdot \left\{ c_1(T_1^4 - T_2^4) \cdot \varphi(x_2) + \int_{A_2} g(x_2')\,\varphi(x_2 x_2')\,dx_2' \right\} \tag{13}$$

$$H = A_1 c_1 (T_1^4 - T_2^4) - \frac{c_1}{c_0} \cdot \int_{A_2} g(x_2)\,\varphi(x_2)\,dx_2. \tag{14}$$

These equations can be solved without further approximations if the outer surface is a sphere. If it has the radius R, it is seen that

$$\varphi(x_2 x_2') = \frac{\cos i_2 \cdot \cos i_2'}{\pi \cdot r^2} = \frac{1}{4\pi R^2} = \frac{1}{A_2}, \tag{15}$$

because $\cos i_2 = \cos i_2' = \frac{r}{2R}$. The last term in equation (13) therefore is a constant so that

$$g(x_2) = \left(1 - \frac{c_2}{c_0}\right) \cdot c_1(T_1^4 - T_2^4) \cdot \{\varphi(x_2) + \text{const.}\}, \tag{16}$$

where the constant is found by inserting (16) into (13). Making use of (7) it is seen that the constant is equal to

$$\frac{c_0}{c_2} \cdot \left(1 - \frac{c_2}{c_0}\right) \cdot \frac{A_1}{A_2}. \tag{16 a}$$

H is then determined by means of (14). We denote mean values over the outer surface by a bar, e. g.

$$\overline{\varphi^2} = \frac{1}{A_2} \cdot \int_{A_2} [\varphi(x_2)]^2 dx_2 \text{ and } \bar{\varphi} = \frac{1}{A_2} \cdot \int_{A_2} \varphi(x_2)\, dx_2 = \frac{A_1}{A_2}$$

according to (7).

The resulting expression for H is

$$H = A_1 c_1 (T_1^4 - T_2^4) \cdot \left\{ 1 - \frac{A_1}{A_2} \cdot c_1 \cdot \left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \left(1 + \frac{c_2}{c_0} \cdot \frac{\overline{\varphi^2} - (\bar{\varphi})^2}{(\bar{\varphi})^2}\right) \right\}. \tag{17}$$

This may be considered as the first two terms of an expansion of H in powers of $\frac{A_1}{A_2}$. We know that Christiansen's formula (2) is correct when φ is constant, i. e. $\overline{\varphi^2} = (\bar{\varphi})^2$. We therefore obtain a better result if we transform the expansion (17) into an expression in which the denominator is expanded in powers of $\frac{A_1}{A_2}$ instead of the nominator. In this way we get from (17)

$$H = \frac{A_1 c_1 (T_1^4 - T_2^4)}{1 + c_1 \left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{A_1}{A_2} \cdot \left(1 + \frac{c_2}{c_0} \cdot \frac{\overline{\varphi^2} - (\bar{\varphi})^2}{(\bar{\varphi})^2}\right)}. \tag{18}$$

It is of course possible to proceed along these lines and first get the term of order $\left(\frac{A_1}{A_2}\right)^2$ by introducing the now determined first order expressions for $I_2(x_2)$ and $I_1(x_1)$ into the integral $\int_{A_1} \cdots dx_1$, in (4 a). In the case of a surrounding sphere this—second—approximation can also be expressed in terms of simple mean values. Usually, however, (18) will be quite a sufficient approximation.

If the outer surface is not a sphere (18) will no longer be a consequent expansion until the first power of $\frac{A_1}{A_2}$ because (16) is not an exact solution of (13). In order to solve equation (13) in this case we make use of the following result from the theory of integral equations[1]:

[1] See, e. g., COURANT and HILBERT: *Methoden der mathematischen Physik*, Vol. 1.

The integral equation

$$g(x_2) = a\varphi(x_2) + b \cdot \int_{A_2} g(x_2') \varphi(x_2 x_2') dx_2', \tag{19}$$

where a and b are constant and the "kernel" $\varphi(x_2 x_2')$ is symmetrical in x_2 and x_2' (as in our case) has the solution

$$g(x_2) = a\left\{\varphi(x_2) + b \cdot \int_{A_2} \sum_{i=0}^{N} \frac{h_i(x_2) \cdot h_i(x_2')}{\lambda_i - b} \cdot \varphi(x_2') dx_2'\right\}. \tag{20}$$

In this expression $h_i(x_2)$ and λ_i (where i covers 0 to N) are the $N+1$ independent eigenfunctions and corresponding eigenvalues of the kernel $\varphi(x_2 x_2')$; they are defined by the statement that they satisfy the homogeneous integral equation:

$$h_i(x_2) = \lambda_i \cdot \int_{A_2} h_i(x_2') \varphi(x_2 x_2') dx_2'; \tag{21}$$

furthermore the eigenfunctions must be normalised and orthogonal, i. e.:

$$\int_{A_2} h_i(x_2) \cdot h_k(x_2) dx_2 = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}. \tag{22}$$

In our case $\int_{A_2} \varphi(x_2 x_2') dx_2' = 1$; it then follows from (21) that there is always a constant eigenfunction $h_0(x_2)$; owing to the normalisation it must have the value $\frac{1}{\sqrt{A_2}}$. The corresponding eigenvalue is $\lambda_0 = 1$ according to (21). In the solution (20) we treat this eigenfunction separately.

From (13), (19), (20), and (14) we get an expression for H similar to (17). Transforming to a form similar to (18), we finally get

$$H = \frac{A_1 c_1 (T_1^4 - T_2^4)}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{A_1}{A_2}\left(1 + \frac{c_2}{c_0} \cdot k\right)} \tag{23}$$

with

$$k = \frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2} + \left(1 - \frac{c_2}{c_0}\right) \cdot \frac{A_2}{(\overline{\varphi})^2} \cdot \sum_{i=1}^{N} \frac{(\overline{h_i \varphi})^2}{\lambda_i - \left(1 - \frac{c_2}{c_0}\right)}. \tag{24}$$

Generally the eigenfunctions cannot be found explicitly. However, it will often be a sufficiently good approximation to neglect the eigenfunctions of higher order than the zero$^{\text{th}}$, because they have zeropoints and consequently give smaller contributions to k than the zero$^{\text{th}}$. Below we shall treat an example where these contributions can be evaluated. If the eigenfunctions of higher order are neglected, (23) and (18) are identical.

Examples.

Two spheres.

We consider two spheres, one inside the other, with radii R and r and placed excentrically with a distance c between the centres (cf. fig. 3). We evaluate the function $\varphi(x_2)$ by means of the following quite general rule: The cone made up by the tangents from x_2 to the inner surface cuts a sphere with radius 1 and centre in x_2 in a certain closed curve; this curve is projected on the tangential plane of the outer surface in x_2; the area enclosed by the projection is $\pi \cdot \varphi(x_2)$. In the case of two spheres the projected curve is an ellipse, the area of which can easily be found by simple geometry. For a point x_2 on the outer sphere with a distance y from the centre of the inner sphere we find (for notation cf. fig. 3):

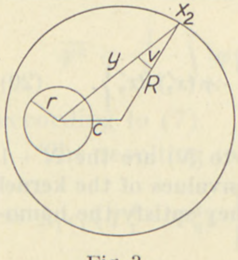

Fig. 3.

$$\varphi(x_2) = \varphi(y) = \frac{r^2}{y^2} \cdot \cos v = \frac{r^2}{y^2} \cdot \frac{R^2 + y^2 - c^2}{2\,Ry}. \tag{25}$$

Integration then leads to

$$k = \frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2} = \frac{-1 + 7\left(\frac{c}{R}\right)^2 - 4\left(\frac{c}{R}\right)^4}{4\left(1 - \left(\frac{c}{R}\right)^2\right)^2} + \frac{1}{8 \cdot \frac{c}{R}} \cdot \log_e \frac{1 + \frac{c}{R}}{1 - \frac{c}{R}}; \tag{26}$$

k is zero when $c = 0$; it is always positive and increases monotonously until $c = R - r$ (the spheres touch each other). The value of k for $c = R - r$—denoted by $k_{\max}$—is given below for some values of $\frac{R}{r}$:

$\frac{R}{r}$	8	4	2	1
$k_{\max}$	10	2.5	0.5	0

When $c \neq 0$, (18) leads to a smaller loss of energy than Christiansen's formula (2). A few examples will show the order of magnitude of the difference. We take $\frac{R}{r} = 4$; the difference between (2) and (18), when the spheres touch each other, is then

$$6.4\,^0/_0 \text{ if } \frac{c_1}{c_0} = 1 \quad \frac{c_2}{c_0} = 0.5$$

$$3.7\,^0/_0 \text{ if } \frac{c_1}{c_0} = 1 \quad \frac{c_2}{c_0} = 0.75$$

$$2.8\,^0/_0 \text{ if } \frac{c_1}{c_0} = 0.75 \quad \frac{c_2}{c_0} = 0.75.$$

In the same three cases the differences between the values given by the uncorrected formula (1) and Christiansen's formula (2) are 5.9 per cent., 2.0 per cent., and 1.5 per cent., respectively. (All numbers are given in per cent. of the uncorrected expression (1).)

Plane disk inside sphere.

We shall evaluate $k = \frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2}$ for a number of different positions and magnitudes of a plane disk inside a sphere in order to be able to estimate k for any position and magnitude of the disk.

In the first case (cf. fig. 4) the disk is circular and placed with its centre in the centre of the sphere.

It can be proved—by integration—that for a point x_2 with polar distance ϑ:

$$\left.\begin{aligned} \varphi(x_2) = \varphi(\vartheta) = \frac{r^2}{\varrho_1 \varrho_2} \cdot \cos\vartheta = \\ = \frac{r^2 \cos\vartheta}{\sqrt{R^4 + r^4 + 2\,r^2 R^2 \cos 2\vartheta}} \end{aligned}\right\} \quad (27)$$

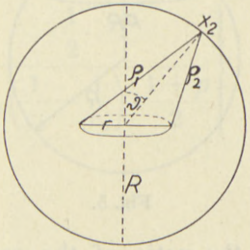

Fig. 4.

(for notation cf. fig. 4).

This leads to

$$k = \frac{R^2 - r^2}{r^2} \cdot \left(1 - \frac{R}{2\,r} \operatorname{arctg} \frac{2\,rR}{R^2 - r^2}\right); \quad (28)$$

k decreases from $\frac{1}{3}$ to 0 when r increases from 0 to R. The decrease is slow when r is small.

Next we consider a—not necessarily circular—disk which is so small, that its magnitude only plays a minor role in determining k. $\left(\text{In the case above this corresponds to } r \text{ being so small that } k \text{ is not far from } \frac{1}{3}\right)$. The disk is situated so that the straight line from the centre of the sphere to it is $p \cdot R$ in length, and the angle between this line and the direction perpendicular to the disk is u. It may then be shown that

$$k = k_1 \cos^2 u + k_2 \sin^2 u, \tag{29}$$

where k_1 and k_2 are the values of k in the two cases where the plate is respectively perpendicular to and parallel with the line to the centre of the sphere. The values of k_1 and k_2 are found by integration:

$$k_1 = \frac{4}{3(1-p^2)^2} - 1 \tag{30}$$

$$k_2 = \frac{1}{4p} \cdot \log_e \frac{1+p}{1-p} + \frac{5-3p^2}{6(1-p^2)^2} - 1. \tag{31}$$

Finally we consider a disk covering a parallel circle, the centre of which has a distance $p \cdot R$ from the centre of the sphere (fig. 5). For a point x_2 on the smaller of the two spherical caps we have:

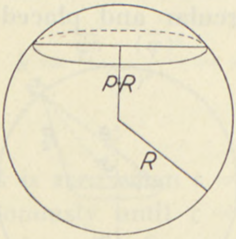

Fig. 5.

$$\left.\begin{aligned} \varphi(x_2) &= \int_{\text{disk}} \varphi(x_2 x_1)\, dx_1 = \int_{\text{big spherical cap}} \varphi(x_2 x_2')\, dx_2' \\ &= \frac{1}{4\pi R^2} \int_{\text{big spherical cap}} dx_2' \qquad = \frac{1+p}{2}. \end{aligned}\right\} \tag{32}$$

In the same way it is seen that $\varphi(x_2)$ is constant on the big spherical cap and equals the ratio of the area of the small spherical cap to the area of the whole sphere, i.e. $\frac{1-p}{2}$. Consequently we find

$$k = \frac{p^2}{1-p^2}. \tag{33}$$

By means of the results from the special cases treated above the value of k may be estimated in most cases without further integration. As an example we consider a circular disk perpendicular to the line from its centre to the centre of the sphere at a distance of $\frac{3}{4}R$ from the centre of the sphere $\left(p = \frac{3}{4}\right)$. If the disk is small, $k \simeq 6$ according to (30), while $k = 1.3$ for the corresponding parallel circle. For values of its radius between zero and the radius of the parallel circle $\left(\frac{\sqrt{7}}{4}R\right)$ we may estimate k from the way in which it is known to vary with radius in the first special case above (formula 28). If the radius is $\frac{1}{4}R$ we find $k \simeq 5$. The difference between the values of H given by (2) and (18), respectively, is—for the three sets of values of $\frac{c_1}{c_0}$ and $\frac{c_2}{c_0}$ used on page 11—: 7 per cent., 4 per cent. and 3 per cent., while the correction in Christiansen's formula ((1)—(2)) amounts to 3 per cent., 1 per cent. and 0.8 per cent. in the corresponding cases.

In the case of a parallel circle in a sphere treated above, H can be calculated exactly as a sum of two terms, each of the form (2), because $\varphi(x_2)$ is constant on each of the two spherical caps. We will therefore use this case for an estimation of the magnitude of the error made, when formula (18) is used:

The exact value of H is:

$$H = A_1 c_1 (T_1^4 - T_2^4) \cdot \left\{ \frac{\frac{1}{2}}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{1-p}{2}} + \frac{\frac{1}{2}}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{1+p}{2}} \right\}. \quad (34)$$

The approximation (18) leads to

$$H = A_1 c_1 (T_1^4 - T_2^4) \cdot \frac{1}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{1-p^2}{2}\left(1 + \frac{c_2}{c_0} \cdot \frac{p^2}{1-p^2}\right)}. \quad (35)$$

Christiansen's formula (2) gives

$$H = A_1 c_1 (T_1^4 - T_2^4) \cdot \frac{1}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{1-p^2}{2}}. \quad (36)$$

With the usual three sets of values of $\frac{c_1}{c_0}$ and $\frac{c_2}{c_0}$ (page 11) we find in this special case:

The error in the formula (1) is approximately 30 per cent., 14 per cent. and 11 per cent.
— — - Christiansen's formula (2) is approximately 17 p^2 per cent., 11 p^2 per cent. and 9 p^2 per cent.
— — - the formula (18) is approximately 4 p^2 per cent., 1.3 p^2 per cent. and 1.4 p^2 per cent.

(All numbers in percentages of the simple expression (1).)

Two infinite circular cylinders.

In this case the formulae (23) and (24) ought to be used. We introduce ordinary cylindrical coordinates θ and z for points on the outer cylinder. $g(x_2)$ is independent of z. Consequently the last term in equation (13) can at once be integrated with respect to z'. The result is:

$$g(\theta) = \left(1-\frac{c_2}{c_0}\right) \cdot \left\{ c_1 (T_1^4 - T_2^4)\, \varphi(\theta) + \int_0^{2\pi} g(\theta') \cdot \frac{1}{4} \cdot \left|\sin\frac{\theta'-\theta}{2}\right| d\theta' \right\}. \quad (37)$$

The corresponding homogeneous integral equation is

$$\left.\begin{aligned} h(\theta) &= \lambda \cdot \int_0^{2\pi} h(\theta') \cdot \frac{1}{4} \cdot \left|\sin\frac{\theta'-\theta}{2}\right| d\theta' \\ &= \lambda \cdot \left\{ -\int_0^{\theta} h(\theta') \cdot \frac{1}{4} \sin\frac{\theta'-\theta}{2}\, d\theta' + \int_\theta^{2\pi} h(\theta') \cdot \frac{1}{4} \sin\frac{\theta'-\theta}{2}\, d\theta' \right\}. \end{aligned}\right\} \quad (38)$$

Differentiation of this equation with respect to θ shows that the eigenfunction $h(\theta)$ must satisfy the differential equation

$$\frac{d^2 h}{d\theta^2} = \frac{1}{4}(\lambda - 1) \cdot h, \quad (39)$$

the solutions of which are $\frac{\sin}{\cos}\left\{\frac{\sqrt{1-\lambda}}{2} \cdot \theta\right\}$.
They must be periodic with period 2π, whence it follows that $\frac{\sqrt{1-\lambda}}{2}$ must be an integer. If we normalise the eigenfunctions according to the rule:

$$\int_0^{2\pi} [h(\theta)]^2 d\theta = 1, \quad (40)$$

we get the following series of independent eigenfunctions:

$$h_p(\theta) = \frac{1}{\sqrt{\pi}} \cdot \begin{Bmatrix} \cos p\theta \\ \sin p\theta \end{Bmatrix} \quad (p = 1, 2, \cdots) \tag{41}$$

with the corresponding eigenvalues

$$\lambda_p = 1 - 4p^2. \qquad (p = 1, 2, \cdots). \tag{42}$$

Besides these, we have the constant eigenfunction

$$h_0 = \frac{1}{\sqrt{2\pi}} \text{ with } \lambda_0 = 1. \tag{43}$$

Inserting these eigenfunctions and eigenvalues in (24) and the resulting k in (23), we find H. (It should be noted that the methods of normalisation used here and on page 9 are different, because A_2 now is infinite). If the zero-plane for θ is taken to be the plane through the axes of the cylinders, $\overline{\sin p\theta \cdot \varphi(\theta)} = 0$ (odd function of θ). Denoting the radii of the inner and outer cylinder by r and R, respectively, we find that the energy loss per unit of length from the inner cylinder is

$$H = \frac{2\pi r c_1 (T_1^4 - T_2^4)}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{r}{R} \cdot \left(1 + \frac{c_2}{c_0} \cdot k\right)} \tag{44}$$

with

$$k = \frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2} - \left(1 - \frac{c_2}{c_0}\right) \cdot \sum_{p=1}^{\infty} \frac{2}{4p^2 - \frac{c_2}{c_0}} \left(\frac{\overline{\varphi \cos p\theta}}{\overline{\varphi}}\right)^2. \tag{45}$$

The distance between the axes of the cylinders we denote by c. The function $\varphi(x_2) = \varphi(\theta)$ may be determined by the method used on page 10 through a rather simple geometrical consideration. With x denoting the distance from the point θ to the axis of the inner cylinder we get:

$$\varphi(\theta) = \frac{r}{2R} \cdot \frac{R^2 + x^2 - c^2}{x^2}. \tag{46}$$

From this we get by integration

$$\frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2} = \frac{c^2}{2(R^2 - c^2)}. \tag{47}$$

The terms in the sum in (45) can be evaluated by integration (most easily by contour integration). We find:

$$\left(\frac{\overline{\varphi \cos p\theta}}{\overline{\varphi}}\right)^2 = \frac{1}{4}\left(\frac{c^2}{R^2}\right)^p. \tag{48}$$

When (47) and (48) are inserted in (45) we get

$$k = \frac{c^2}{2(R^2 - c^2)} - \frac{1}{2} \sum_{p=1}^{\infty} \frac{1 - \frac{c_2}{c_0}}{4p^2 - \frac{c_2}{c_0}} \cdot \left(\frac{c^2}{R^2}\right)^p. \tag{49}$$

It is seen that the terms from the higher eigenfunctions decrease rapidly with increasing order. Even the contribution from the second term in (49) may generally be neglected. If e. g. $\frac{c_2}{c_0} = 0.75$, the ratio between the first and second term in (49) will be $13 \frac{R^2}{R^2 - c^2}$, which is more than 13.

This result gives some justification for totally neglecting contributions from the other eigenfunctions than the zero[th], i. e. for using formula (18) even in cases where the outer surface is not a sphere.

We again calculate the difference between the values of H given by (1) and (2) and by (2) and (18) for the usual three sets of values of $\frac{c_1}{c_0}$ and $\frac{c_2}{c_0}$ (page 11). We choose as an example $r = \frac{1}{8} R$ and $c = R - r$ (the cylinders touch). By using formula (18) we neglect other terms in (49) than the first. (The contribution from the second is in this case 3—2 per cent. of the first). The result is that the difference between Christiansen's formula (2) and our formula (18) is 7.4 per cent., 4.5 per cent. and 3.5 per cent., respectively, while the corresponding differences between (1) and (2) are 11 per cent, 4 per cent. and 3 per cent.

The examples treated above show that in case of a very unsymmetrical position of the inner body with respect to the outer one, it is often so that very little is obtained by applying Christiansen's formula in calculating the heat transfer, because the error made may be just as large or larger than the correction which the formula gives compared with the simple expression $c_1 A_1 (T_1^4 - T_2^4)$. We must conclude that if we aim at such an accuracy that it is necessary to apply a corrected formula instead of (1), then (18) must be used in case of unsymmetrical position of the inner body. This is also practically possible, because the order of magnitude of the correction factor $\frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2}$ can often be estimated by simple geometrical or graphical methods.

Discussion of the assumptions made on page 3.

Equations analogous to (3), (4a), and (5) can easily be obtained in the most general case. The surface in the vicinity of a point x has the absolute temperature T and a reflectivity denoted by $(i\alpha\,|\,r(\lambda\,Tx)\,|\,i'\alpha')$ and defined in the following way:

We consider monochromatic radiation of wavelength λ, which is falling on a surface element dA. The direction of the incoming ray will be characterized by the angles i (angle of incidence) and α (azimuth) as shown on fig. 6. Of this radiation a certain fraction will be reflected so that it leaves the surface within a solid angle $d\omega'$, the principal direction of which is characterized by i' and α' (cf. fig. 6). If the reflection is completely diffuse the reflected radiation is distributed according to the cosine law, i. e. the said fraction will be proportional to $\cos i'$ and independent of i, α, and α'. In general we therefore denote the fraction reflected to $d\omega'$ by

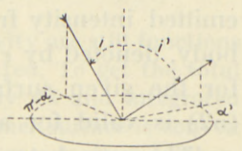

Fig. 6.

$$\frac{1}{\pi}\cdot(i\alpha\,|\,r(\lambda\,Tx)\,|\,i'\alpha')\cdot\cos i'\,d\omega'. \tag{50}$$

The notation $r(\lambda\,Tx)$ is chosen in order to show that the reflectivity in general will depend on wavelength, temperature, and constitution of the surface as well as on the angles. The factor $\frac{1}{\pi}$ is introduced for convenience.

The intensity of the emitted radiation (emitted radiation from unit of apparent area into unit of solid angle) in an arbitrary direction can be calculated by means of Kirchhoff's law, if the reflectivity is known:

We shall write down an equation expressing that the resulting intensity of radiation in a closed cavity, the walls of which all have the same temperature T, must everywhere be equal to the intensity of radiation from a black body $K_0(\lambda\,T)$. The resulting intensity from a point x in the direction $(i\alpha)$ is the sum of the emitted intensity $K_T(\lambda\,Tx, i\alpha)$ and the reflected intensity, an expression for which can be written down by means of (50); the equation is:

$$K_0(\lambda\,T) = K_T(\lambda\,Tx, i\alpha) + K_0(\lambda\,T)\cdot\frac{1}{\pi}\cdot\int_{\text{solid angle } 2\pi}(i'\alpha'\,|\,r(\lambda\,Tx)\,|\,i\alpha)\cos i'\,d\omega' \tag{51}$$

or

$$e(\lambda Tx, ia) = \frac{K_T(\lambda Tx, ia)}{K_0(\lambda T)} = 1 - \frac{1}{\pi} \cdot \int_{\text{solid angle } 2\pi} (i'a' \,|\, r(\lambda Tx) \,|\, ia) \cos i' d\omega'. \quad (52)$$

The black body intensity $K_0(\lambda T)$ is given by Planck's radiation formula (it is independent of the angles). The ratio between the emitted intensity from the surface in question and from a black body, denoted by $e(\lambda Tx, ia)$ in (52), will be called the emissivity for the given surface, wavelength, temperature, and direction. (52) is valid for all wavelengths.

The ratio between the total hemispherical radiation of wavelength λ emitted from unit of area of the surface in question and from unit of area of a black body is:

$$E(\lambda Tx) = \frac{1}{\pi} \int_{\text{solid angle } 2\pi} e(\lambda Tx, ia) \cos i \, d\omega. \quad (53)$$

(For the so called "grey" surfaces treated above we have the equalities:

$$e(\lambda Tx, ia) = E(\lambda Tx) = \frac{c}{c_0}\Big).$$

We further define the absorptivity $a(\lambda Tx, ia)$, i. e. the fraction absorbed of radiation coming in from the direction (ia):

$$a(\lambda Tx, ia) = 1 - \frac{1}{\pi} \int_{\text{solid angle } 2\pi} (ia \,|\, r(\lambda Tx) \,|\, i'a') \cos i' d\omega'. \quad (54)$$ [1]

We now get the equations analogous to (3) and (4a) by expressing that the resulting intensity of radiation $K(iax)$ emerging from the point x on one of the surfaces in the direction (ia) is the sum of the emitted and reflected intensities:

$$K_1(i_1 a_1 x_1) = K_{T_1}(i_1 a_1) + \int_{A_2'} (i_1' a_1' \,|\, r_1 \,|\, i_1 a_1) K_2(i_2 a_2 x_2) \cdot \varphi(x_1 x_2) \, dx_2. \quad (55)$$

$$\left.\begin{aligned} K_2(i_2 a_2 x_2) = K_{T_2}(i_2 a_2) + \int_{A_1'} (i_2' a_2' \,|\, r_2 \,|\, i_2 a_2) \cdot \left[K_1(i_1 a_1 x_1) - K_2(i_2'' a_2'' X_2)\right] \cdot \\ \cdot \varphi(x_1 x_2) \, dx_1 + \int_{A_2} (i_2' a_2' \,|\, r_2 \,|\, i_2 a_2) \cdot K_2(i_2'' a_2'' x_2') \, \varphi(x_2 x_2') \, dx_2'. \end{aligned}\right\} \quad (56)$$

[1] If the so called Helmholtz's reciprocity law (H. v. Helmholtz: *Theoretische Physik*, Vol. 6, p. 161, 1903) is valid, we have: $(ia \,|\, r \,|\, i'a') = (i'a' \,|\, r \,|\, ia)$, and consequently $e \equiv a$.

The heat transfer from the inner body is in analogy with (5) (for monochromatic radiation):

$$H = A_1 \pi \cdot E_1 \cdot K_0(T_1) - \pi \cdot \int_{A_1} dx_1 \int_{A_2'} K_2(i_2 a_2 x_2) \cdot a_1(i_1 a_1) \cdot \varphi(x_1 x_2)\, dx_2 . \quad (57)$$

(The emitted intensity of radiation and the reflectivity are still functions of $(\lambda T x)$, although these variables have been omitted. To get the total radiation the equation (57) must be multiplied by $d\lambda$ and integrated over all wavelengths. It is still assumed that the inner surface is convex and the outer one concave, and that both bodies are opaque; furthermore the inner body must have the same temperature and emissivity everywhere if (57) is to be correct. Apart from this the equations are quite general.)

Of course the equations can only be solved exactly in special cases, of which we are going to consider some in what follows, in order to exemplify the applicability of the method of integral equations.

(a) Non-validity of Stefan-Boltzmann's law.

If the reflectivity depends on wavelength and temperature, but not on angles and position (x), the assumptions (3) on page 3 are not valid, but the rest is. In this case all the calculations in the first part of this paper hold true for the heat transfer caused by radiation in a narrow interval $d\lambda$ of wavelength (monochromatic radiation). (Fluorescence, etc., must of course be excluded.) The total heat transfer is then obtained by integration over all wavelengths. The formula analogous to (18) now is

$$H = \int_0^\infty d\lambda \cdot \left\{ \frac{\pi \cdot A_1 \cdot E_1(\lambda T_1)\,[K_0(\lambda T_1) - K_0(\lambda T_2)]}{1 + \frac{E_1(\lambda T_1)}{E_2(\lambda T_2)} \cdot (1 - E_2(\lambda T_2)) \cdot \frac{A_1}{A_2} \cdot \left(1 + E_2(\lambda T_2) \cdot \frac{\overline{\varphi^2} - (\overline{\varphi})^2}{(\overline{\varphi})^2}\right)} \right\} . \quad (58)$$

The ratios $\frac{c_1}{c_0}$ and $\frac{c_2}{c_0}$ from the case of "grey" bodies have been replaced by $E_1(\lambda T_1)$ and $E_2(\lambda T_2)$, while $c_0 T^4$ has been replaced by $\pi K_0(\lambda T)$.

(b) Non-validity of the cosine law for the inner surface.

The inner surface is now assumed to reflect in an arbitrary way, while the outer one still has a reflectivity that is independent of the angles (completely diffuse reflection). It is then easy to show from (55), (56), and (57), in a way similar to that which led to formula (18) or (58), that formulae analogous to (18) or (58) hold, but the term $\frac{\overline{\varphi^2}-(\overline{\varphi})^2}{(\overline{\varphi})^2}$ must be replaced by

$$k' = \frac{\overline{\psi\psi'}-(\overline{\psi})^2}{(\overline{\psi})^2}, \tag{59}$$

where

$$\psi(x_2) = \int_{A_1'} \frac{e_1(\lambda T_1 i_1 a_1)}{E_1(\lambda T_1)} \cdot \varphi(x_1 x_2)\, dx_1 \tag{60}$$

and

$$\psi'(x_2) = \int_{A_1'} \frac{a_1(\lambda T_1 i_1 a_1)}{E_1(\lambda T_1)} \cdot \varphi(x_1 x_2)\, dx_1\,; \qquad (61)^1$$

ψ and ψ' are straightforward generalisations of the function $\varphi(x_2)$ for emission and reflection, respectively. The definitions (52), (53), and (54) together with the definition of $\varphi(x_1 x_2)$ show that

$$\overline{\psi(x_2)} = \overline{\psi'(x_2)} = \overline{\varphi(x_2)} = \frac{A_1}{A_2}. \tag{62}$$

It is worth noticing that Christiansen's formula (2), perhaps modified in order to take into account a possible dependence of the reflectivity on wavelength and temperature, still holds for concentric spheres and coaxial cylinders, in which cases $\psi(x_2)$ and $\psi'(x_2)$ are constant. But $\psi(x_2)$ and $\psi'(x_2)$ are not necessarily constant in all cases where $\varphi(x_2)$ is so. They are not so, e. g., in the case treated on page 6 and 12, where the outer surface is a sphere and the inner body a disk covering the equatorial plane. In such cases Christiansen's formula therefore is only correct, if the inner body radiates according to the cosine law.

(c) Non-validity of the cosine law for the reflection from the outer surface.

The case which gives the largest deviation from the uncorrected formula (1) is the following: The system consists of two concentric spheres or coaxial cylinders of which the outer is reflecting specularly, Every ray which, coming from the inner

[1] If Helmholtz's reciprocity-law holds, then $\psi(x_2) = \psi'(x_2)$.

surface, is reflected on the outer one, will then hit the inner surface again. If especially the reflectivities are independent of wavelength and temperature, and the absorptivities are independent of the angles, the reduction in loss of energy due to reflection must therefore be the same as if the inner body was closely surrounded by the outer one. We then get in place of (2)

$$H = \frac{A_1 \cdot c_1 \cdot (T_1^4 - T_2^4)}{1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right)}. \tag{63}$$

This formula is also due to Christiansen (footnote on page 4). It can easily be generalised to the case of wavelength- and temperature-dependent reflectivities.

As mentioned above, (63) is valid for concentric spheres and coaxial cylinders. But as soon as the spheres or cylinders are placed a little excentrically, or deformed somehow, the fraction of the reflected radiation that reenters on the inner surface will decrease considerably, and the loss of energy increases. It is worth noticing that the loss of energy in case of specular reflection is the smallest possible in the concentric position, while it is largest in this position if the reflection is diffuse. The formulae (2) and (63) give the maximum and minimum values of the loss of energy, while excentric position or unsymmetric form gives formulae like (18) lying between (2) and (63).

It will often be a good approximation to assume that the outer surface is reflecting a certain fraction s of the reflected radiation completely diffusely, while the rest — $(1 - s)$ — is reflected specularly. Furthermore we assume that s and the total reflectivity is independent of the angle of incidence. The heat transfer between concentric spheres or coaxial cylinders can then easily be calculated. Either (55), (56), and (57) may be used (for the outer surface we may put $(ia \,|\, r_2 \,|\, i'a') = \left(1 - \frac{c_2}{c_0}\right) \cdot \left\{s + \frac{\pi \cdot (1-s)}{\sin i \cdot \cos i} \cdot \delta(i - i') \cdot \delta(a - (a' + \pi))\right\}$, where $\delta(x - x')$ is Dirac's δ-function), or we may at once write down analogous equations for the total resulting radiation from the two surfaces. We only give the result in case the radiation constants are independent of wavelength and temperature and the absorptivity of the inner surface is independent of the angles:

$$H = A_1 c_1 (T_1^4 - T_2^4) \cdot \frac{s + (1-s) \cdot \frac{c_2}{c_0}}{s\left[1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right) \cdot \frac{A_1}{A_2}\right] + (1-s) \cdot \frac{c_2}{c_0} \cdot \left[1 + c_1\left(\frac{1}{c_2} - \frac{1}{c_0}\right)\right]}. \quad (64)$$

This shows that the heat transfer is nearer to the value for diffuse reflection, i. e. larger, than that obtained by simply adding expressions of the form (2) and (63) in the ratio s: $(1-s)$.

It has not been found possible to obtain a general formula in case of angle-dependent reflectivity of the outer surface.

(d) The temperature and emissivity of the outer body varies over the surface. Resulting radiation field within a closed cavity.

Let all assumptions made on page 3 be correct, except that T_2 and c_2 are functions of x_2.

We only treat the case of a very small inner body $\left(\frac{A_1}{A_2} \simeq 0\right)$ corresponding to the "zero$^{\text{th}}$" approximation (1). Consequently we must solve the equation for $I_2(x_2)$ without contributions from an inner body. For convenience we omit the index 2 in I_2, x_2, c_2, etc.:

$$I(x) = c(x) \cdot T(x)^4 + \left(1 - \frac{c(x)}{c_0}\right) \cdot \int_A I(x') \cdot \varphi(xx')\, dx'. \quad (65)$$

This equation is a straightforward generalisation of (4a) or specialisation of (56). If T is a constant, it has of course the solution $I(x) = c_0 T^4$ irrespective of the values of $c(x)$ and the form of the enclosure (black body radiation). In general, however, it can only be solved numerically, e. g. by replacing it by a number of linear equations corresponding to the required accuracy. If the cavity is a sphere, it can be solved exactly, for in this case we have $\varphi(xx') = \frac{1}{A}$, so that (65) takes the form

$$I(x) = c(x) \cdot T(x)^4 + \left(1 - \frac{c(x)}{c_0}\right) \cdot \bar{I} \quad (66)$$

where $\overline{I} = \frac{1}{A}\int_A I(x')\,dx'$ is the mean value of $I(x)$ over the surface. Taking mean values of the terms in (66), we get

$$\overline{I} = \overline{cT^4} + \left(1 - \frac{\overline{c}}{c_0}\right)\cdot \overline{I}, \tag{67}$$

whence

$$\overline{I} = c_0 \cdot \frac{\overline{cT^4}}{\overline{c}}, \tag{68}$$

which inserted in (66) leads to

$$I(x) = c_0 \cdot \frac{\overline{cT^4}}{\overline{c}} + c(x)\cdot\left[T(x)^4 - \frac{\overline{cT^4}}{\overline{c}}\right]. \tag{69}$$

From this result the heat exchange with a small body with uniform temperature T_1 and radiation constant c_1 can be calculated when the contributions from this body to the radiation field can be neglected $\left(\frac{A_1}{A_2} \sim 0\right)$. By means of (69) and (5) page 5 we get instead of (1):

$$\left.\begin{aligned} H &= A_1 c_1\left[T_1^4 - \frac{1}{c_0}\cdot\frac{\overline{I(x)\cdot\varphi(x)}}{\overline{\varphi(x)}}\right] = \\ &= A_1\cdot c_1\cdot\left[T_1^4 - \frac{\overline{cT^4}}{\overline{c}} - \frac{1}{c_0}\frac{\overline{cT^4\varphi}}{\overline{\varphi}} + \frac{1}{c_0}\cdot\frac{\overline{cT^4}}{\overline{c}}\cdot\frac{\overline{c\varphi}}{\overline{\varphi}}\right].\end{aligned}\right\} \tag{70}$$

We may define a "resulting radiation temperature" T_0 of the sphere with respect to the small inner body as the uniform temperature which the sphere ought to have if it were black and were to exchange the same amount of heat as (70) with the inner body, i. e. we put

$$H_1 = A_1 c_1 (T_1^4 - T_0^4), \tag{71}$$

whence

$$T_0^4 = \frac{1}{c_0}\cdot\frac{\overline{I\varphi}}{\overline{\varphi}} = \frac{\overline{cT^4}}{\overline{c}} + \frac{1}{c_0}\cdot\frac{\overline{cT^4\varphi}}{\overline{\varphi}} - \frac{1}{c_0}\cdot\frac{\overline{cT^4}}{\overline{c}}\cdot\frac{\overline{c\varphi}}{\overline{\varphi}}. \tag{72}$$

This is strictly correct for a sphere and will probably be a good approximation in many other cases. If the temperature does not vary too much, it may be a sufficient approximation to use the temperatures in °C in (72) instead of the fourth powers of the absolute temperatures.

(e) Cavities in a surface.

A cavity the walls of which have a certain emissivity, may be replaced by a surface covering the cavity, but with another emissivity, which generally will vary over the surface and depend on the direction of emission. The method of integral equations can also be used to find this apparent emissivity.

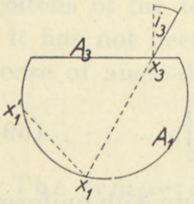

Fig. 7.

We assume that the walls of the cavity are reflecting diffusely and independently of wavelength, temperature, and direction. The radiation constant is c_1 and the uniform temperature T_1; the emitted intensity of radiation is then $\frac{1}{\pi} c_1 T_1^4$ in all directions. We are going to find the resulting radiation intensity $K_3(x_3 i_3 a_3)$ in an arbitrary point x_3 on the replacing surface A_3 and in an arbitrary direction (i_3, a_3) (cf. fig. 7). First, we have that the intensity sought for is equal to the resulting intensity from the corresponding point x_1 (cf. fig. 7):

$$K_3(x_3 i_3 a_3) = K_1(x_1) \tag{73}$$

$K_1(x_1)$ is independent of direction owing to the completely diffuse reflection and may be found from an integral equation expressing it as a sum of emitted and reflected radiation as usual:

$$K_1(x_1) = \frac{1}{\pi} c_1 T_1^4 + \left(1 - \frac{c_1}{c_0}\right) \cdot \int_{A_1} K_1(x_1') \cdot \varphi(x_1 x_1')\, dx_1'. \tag{74}$$

It is seen that only if the function

$$\varphi(x_1) = \int_{A_1} \varphi(x_1 x_1')\, dx_1' = 1 - \int_{A_3} \varphi(x_1 x_3)\, dx_3 \tag{75}$$

is independent of x_1, we find a constant value of $K_1(x_1)$, and only in this case, therefore, K_3 is independent of x_3 and the direction $(i_3 a_3)$.

If $\varphi(x_1)$ is constant, the value of it may be found, because

$$\int_{A_1} dx_1 \int_{A_3} \varphi(x_1 x_3)\, dx_3 = \int_{A_3} dx_3 \int_{A_1} \varphi(x_1 x_3)\, dx_1 = A_3. \tag{76}$$

From this and (75) it follows that

$$\varphi(x_1) = 1 - \frac{A_3}{A_1}. \tag{77}$$

(In (76) it is assumed, that A_3 is plane.)
With this value of $\varphi(x_1)$, K_3 may be found from (73) and (74):

$$K_3 = K_1 = \frac{\frac{1}{\pi}\cdot c_1 \cdot T_1^4}{1 - \left(1 - \frac{c_1}{c_0}\right)\left(1 - \frac{A_3}{A_1}\right)}. \tag{78}$$

We define the apparent radiation constant c_3 by putting

$$K_3 = \frac{1}{\pi} c_3 T_1^4. \tag{79}$$

(79) and (78) then lead to

$$\frac{1}{c_3} = \frac{1}{c_1}\cdot\frac{A_3}{A_1} + \frac{1}{c_0}\left(1 - \frac{A_3}{A_1}\right). \tag{80}$$

It is seen that $c_3 \to c_0$, if $\frac{A_3}{A_1} \to 0$, as it must, because we then get an artificial "black body".

If $\varphi(x_1)$ is not constant, (74) may be solved numerically or by iteration. $\varphi(x_1)$ is constant, if the cavity is a spherical cap (cf. page 12). If the distance from the centre of the sphere with radius R to the plane A_3 is $p \cdot R$ (p positive to the interior of the cavity), we get $\frac{A_3}{A_1} = \frac{1+p}{2}$, whence

$$\frac{1}{c_3} = \frac{1}{c_1}\cdot\frac{1+p}{2} + \frac{1}{c_0}\cdot\frac{1-p}{2} \qquad (-1 < p < 1). \tag{81}$$

The results and methods used in this section and section *d* may be useful in estimating the deviations of the radiation from a cavity from black body radiation.

Summary. The net loss of energy suffered by a radiating body entirely surrounded by another body of different temperature is investigated with special respect to its dependence on the form and mutual position of the bodies. Integral equations are given which determine the heat transfer ((3), (4a), and (5) for

"grey" radiation, and (55), (56), and (57) in the general case). The equations for "grey" radiation are solved approximately and a formula for the heat transfer is given — (18) — and applied to several examples. The radiation between surfaces which are not grey is treated in some special cases. On page 22 (section *d*) the case of variation in temperature on the outer body is treated, and formulae for the radiation field inside a sphere and for the heat exchange with a small body inside a sphere are obtained (formulae (69) — (72)). Finally, in section (*e*), page 24, equations determining the apparent emissivity of a cavity are obtained and solved for a cavity shaped as a spherical cap.

The methods and results may be of some interest in the heating technique, the illumination technique, and optical pyrometry.

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